

# A reproducing kernel for nonsymmetric Macdonald polynomials

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ABSTRACT. We present a new formula of Cauchy type for the nonsymmetric Macdonald polynomials which are joint eigenfunctions of  $q$ -Dunkl operators. This gives the explicit formula for a reproducing kernel on the polynomial ring of  $n$  variables.

## §0: Introduction.

In this paper we propose a new formula of Cauchy type for the nonsymmetric Macdonald polynomials of type  $A_{n-1}$ . This can be regarded as an explicit formula for the reproducing kernel of a certain scalar product on the polynomial ring of  $n$  variables. A similar result for nonsymmetric Jack polynomials was recently given by Sahi [S].

The nonsymmetric Macdonald polynomials  $E_\lambda(x|q, t)$ , introduced by Macdonald [Ma1], are characterized as the joint eigenfunctions in the polynomial ring of  $n$  variables  $x = (x_1, \dots, x_n)$ , for the commuting family of  $q$ -Dunkl operators. (For the precise definition of  $E_\lambda(x|q, t)$ , see Section 1.) We define a meromorphic function  $E(x; y|q, t)$  in  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  by

$$(0.1) \quad E(x; y|q, t) = \prod_{1 \leq j < i \leq n} \frac{(qt x_i y_j; q)_\infty}{(q x_i y_j; q)_\infty} \prod_{1 \leq i \leq n} \frac{(q t x_i y_i; q)_\infty}{(x_i y_i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}.$$

The main result of this paper is the following.

**Theorem A.** *The function  $E(x; y|q, t)$  has the following expansion in terms of nonsymmetric Macdonald polynomials:*

$$(0.2) \quad E(x; y|q, t) = \sum_{\lambda \in \mathbb{N}^n} a_\lambda(q, t) E_\lambda(x|q, t) E_\lambda(y|q^{-1}, t^{-1}).$$

For each composition  $\lambda \in \mathbb{N}^n$ , the coefficient  $a_\lambda(q, t)$  is given by

$$(0.3) \quad a_\lambda(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}},$$

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where, for each box  $s \in \lambda$ ,  $a(s)$  and  $l(s)$  are the arm-length and the generalized leg-length of  $s$  in  $\lambda$ .

(See Theorems 2.1 and 2.2.)

Assuming that  $q$  and  $t$  are complex numbers with  $0 < |q|, |t| < 1$ , we now consider the meromorphic function  $K(x; y|q, t) = E(x; y^{-1}|q, t)$  on the algebraic torus  $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ .

**Theorem B.** *For each composition  $\lambda \in \mathbb{N}^n$ , we have*

$$(0.4) \quad \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} K(x; y|q, t) E_\lambda(y|q, t) w(y|q, t) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} = C_\lambda \cdot E_\lambda(x|q, t)$$

for all  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$  with  $|x_j| < 1$  ( $j = 1, \dots, n$ ). Here  $\mathbb{T}^n = \{y = (y_1, \dots, y_n) \in (\mathbb{C}^*)^n; |y_j| = 1 \text{ } (j = 1, \dots, n)\}$  is the  $n$ -dimensional torus with the standard orientation, and

$$(0.5) \quad w(y|q, t) = \prod_{1 \leq i < j \leq n} \frac{(y_i/y_j; q)_\infty (qy_j/y_i; q)_\infty}{(ty_i/y_j; q)_\infty (qty_j/y_i; q)_\infty}.$$

The constant  $C_\lambda$  is given by

$$(0.6) \quad C_\lambda = C_{\lambda^+} = \left( \frac{(qt; q)_\infty}{(q; q)_\infty} \right)^n \prod_{i=1}^n \frac{(q^{\lambda_i^+ + 1} t^{n-i}; q)_\infty}{(q^{\lambda_i^+ + 1} t^{n-i+1}; q)_\infty},$$

where  $\lambda^+$  is the partition obtained by reordering the parts of  $\lambda$ .

(See Theorem 3.2.)

After preliminaries on nonsymmetric Macdonald polynomials, we will state our main results in Section 2. In Section 2, we will prove that  $E(x; y|q, t)$  has an expansion of the form (0.2), and reduce the determination of the coefficients  $a_\lambda(q, t)$  to the case of partitions. In this paper we will present two ways of determining the coefficients  $a_\lambda(q, t)$  for partitions  $\lambda$ . In Section 3, we determine these coefficients in an analytic way by asymptotic analysis of  $q$ -Selberg type integrals similarly as in [Mi2]. In this proof we will make use of the evaluation of Cherednik's scalar product for nonsymmetric Macdonald polynomials. Theorem B will also be formulated in Section 3. In Section 4, we give an algebraic proof of (0.3) by using the evaluation of the nonsymmetric Macdonald polynomials  $E_\lambda(x; q, t)$  at  $x = (t^{-n+1}, t^{-n+2}, \dots, 1)$  due to Cherednik [C2]. This second proof is an extension of the argument of Sahi [S] to the  $q$ -version.

## §1: Nonsymmetric Macdonald polynomials.

We first recall the definition of *nonsymmetric Macdonald polynomials* of type  $A_{n-1}$  in the  $GL_n$  version. Although we follow the presentation by Macdonald [Ma2] in principle, we use a slightly different convention which is more convenient for our purpose.

Let  $\mathbb{K}[x^{\pm 1}]$  be the ring of Laurent polynomials in  $n$  variables  $x = (x_1, \dots, x_n)$  with coefficients in the field  $\mathbb{K} = \mathbb{Q}(q, t^{\frac{1}{2}})$ . (Although we use this coefficient field for convenience, the use of  $t^{\frac{1}{2}}$  is *not* essential; one could work within  $\mathbb{Q}(q, t)$  by modifying the argument appropriately. ) We denote by  $\tau = (\tau_1, \dots, \tau_n)$  the corresponding  $q$ -shift operators. For each  $i = 1, \dots, n$ , the operator  $\tau_i$  acts on  $\mathbb{K}[x^{\pm 1}]$  as a  $\mathbb{K}$ -automorphism such that  $\tau_i(x_j) = x_j q^{\delta_{ij}}$  ( $j = 1, \dots, n$ ). The action of the symmetric group  $W = \mathfrak{S}_n$  on  $\mathbb{K}[x^{\pm 1}]$  will be fixed so that each  $w \in W$  defines the  $\mathbb{K}$ -algebra automorphism such that  $w.x_j = x_{w(j)}$  for  $j = 1, \dots, n$ . The ring  $\mathbb{K}[x^{\pm 1}]$  is identified with the group-ring  $\mathbb{K}[P]$  of the integral weight lattice  $P = \mathbb{Z}\epsilon_1 \oplus \dots \oplus \mathbb{Z}\epsilon_n$ . As usual, we take the symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $P$  such that  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$  ( $1 \leq i, j \leq n$ ). For each  $\lambda \in P$ , we use the notation of multi-indices

$$(1.1) \quad x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}, \quad \tau^\lambda = \tau_1^{\lambda_1} \dots \tau_n^{\lambda_n},$$

where  $\lambda_j = \langle \lambda, \epsilon_j \rangle$  ( $j = 1, \dots, n$ ). The action of the symmetric group  $W = \mathfrak{S}_n$  of degree  $n$  on  $P$  will be fixed as  $w.\epsilon_i = \epsilon_{w(i)}$ , or equivalently as  $(w.\lambda)_i = \lambda_{w^{-1}(i)}$  ( $i = 1, \dots, n$ ) for each  $w \in W$ . Note that the commutation relations among the multiplication operators  $x^\lambda$ , the  $q$ -shift operators  $\tau^\mu$  and the permutations  $w \in W$  are given as follows:

$$(1.2) \quad \tau^\mu x^\lambda = x^\lambda \tau^\mu q^{\langle \mu, \lambda \rangle}, \quad wx^\lambda = x^{w.\lambda} w, \quad w\tau^\mu = \tau^{w.\mu} w,$$

for  $\lambda, \mu \in P$  and  $w \in W$ . We will use the standard notation of the set of positive roots

$$(1.3) \quad \Delta^+ = \{\epsilon_i - \epsilon_j ; 1 \leq i < j \leq n\},$$

the simple roots  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  ( $i = 1, \dots, n-1$ ) and the cone of dominant integral weights

$$(1.4) \quad P^+ = \{\lambda \in P ; \langle \alpha_i, \lambda \rangle \geq 0 \ (i = 1, \dots, n-1)\} = \{\lambda \in P ; \lambda_1 \geq \dots \geq \lambda_n\}.$$

We denote the set of *compositions* and the set of *partitions* with length  $\leq n$  by  $L = \mathbb{N}\epsilon_1 \oplus \dots \oplus \mathbb{N}\epsilon_n \subset P$  and by  $L^+ = P^+ \cap L$ , respectively, where  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

In what follows, we will make use of the *Demazure-Lusztig operators*  $T_1, \dots, T_{n-1}$  defined by

$$(1.5) \quad T_i = t^{\frac{1}{2}} + t^{-\frac{1}{2}} \frac{1 - tx_i/x_{i+1}}{1 - x_i/x_{i+1}} (s_i - 1) \quad (i = 1, \dots, n-1),$$

where  $s_i = (i, i+1)$  stands for the reflection with respect to the simple root  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ . Note that

$$(1.6) \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0 \quad (i = 1, \dots, n-1),$$

and that the operators  $T_1, \dots, T_{n-1}$  satisfy the fundamental relations for the canonical generators of the Hecke algebra  $H(\mathfrak{S}_n)$ . Furthermore we define the  *$q$ -Dunkl operators*  $Y_1, \dots, Y_n$ , due to Cherednik, by

$$(1.7) \quad Y_i = T_i T_{i+1} \dots T_{n-1} \omega T_1^{-1} \dots T_{i-1}^{-1} \quad (i = 1, \dots, n),$$

where

$$(1.8) \quad \omega = s_{n-1} \cdots s_1 \tau_1 = \dots = \tau_n s_{n-1} \cdots s_1.$$

These operators  $Y_1, \dots, Y_n$  commute with each other and, for any *symmetric* Laurent polynomial  $f(Y)$  of  $Y = (Y_1, \dots, Y_n)$ , Macdonald's symmetric polynomials  $P_\lambda(x) = P_\lambda(x|q, t)$  ( $\lambda \in P^+$ ) [Ma2] satisfy the equation

$$(1.9) \quad f(Y)P_\lambda(x) = P_\lambda(x)f(q^\lambda t^\rho),$$

where  $f(q^\lambda t^\rho)$  denotes the evaluation of  $f$  at the point  $q^\lambda t^\rho = (q^{\lambda_1} t^{\rho_1}, \dots, q^{\lambda_n} t^{\rho_n})$ , and

$$(1.10) \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \frac{1}{2} \sum_{i=1}^n (n - 2i + 1) \epsilon_i.$$

One important fact is that the  $q$ -Dunkl operators have the triangularity with respect to a certain partial ordering of monomials. We define the partial ordering  $\preceq$  in  $P$  as follows: For  $\lambda, \mu \in P$ ,

$$(1.11) \quad \mu \preceq \lambda \quad \Leftrightarrow \quad \mu^+ < \lambda^+ \quad \text{or} \quad (\mu^+ = \lambda^+, \quad \mu \leq \lambda),$$

where  $\lambda^+$  stands for the unique dominant integral weight in the  $W$ -orbit  $W\lambda$  of  $\lambda$  and  $\leq$  is the dominance order ( $\mu \leq \lambda$  means that  $\lambda - \mu$  is a linear combination of positive roots with coefficients in  $\mathbb{N}$ ). Then it turns out that, for any  $\lambda, \mu \in P$ , one has

$$(1.12) \quad Y^\mu x^\lambda = x^\lambda q^{\langle \mu, \lambda \rangle} t^{\langle \mu, \rho(\lambda) \rangle} + (\text{lower order terms under } \preceq),$$

where

$$(1.13) \quad \rho(\lambda) = \frac{1}{2} \sum_{\alpha \in \Delta^+} \varepsilon(\langle \alpha, \lambda \rangle) \alpha,$$

where  $\varepsilon(u) = 1$  if  $u \geq 0$  and  $\varepsilon(u) = -1$  if  $u < 0$ . Note that  $\rho(\lambda)$  is precisely the element  $w_\lambda \cdot \rho$  in the  $W$ -orbit of  $\rho$  if one take the  $w_\lambda$  which has the minimal length among all  $w \in W$  such that  $\lambda = w \cdot \lambda^+$ .

*Remark 1.1.* Because of the definition of  $q$ -Dunkl operators mentioned above, the partial ordering  $\preceq$  and the function  $\varepsilon(u)$  is different from those in [Ma2]. Note that, under our definition of  $\preceq$ , the dominant weight  $\lambda^+$  is maximal in the  $W$ -orbit  $W\lambda$ .

By the triangularity of  $q$ -Dunkl operators mentioned above, one can show that, for each  $\lambda \in P$ , there exists a unique Laurent polynomial  $E_\lambda(x) = E_\lambda(x|q, t)$  such that

$$(1.14) \quad E_\lambda(x) = x^\lambda + (\text{lower order terms under } \preceq),$$

and that

$$(1.15) \quad Y^\mu E_\lambda(x) = E_\lambda(x) q^{\langle \mu, \lambda \rangle} t^{\langle \mu, \rho(\lambda) \rangle}$$

for any  $\mu \in P$ . These Laurent polynomials  $E_\lambda(x) = E_\lambda(x|q, t)$  are called the *nonsymmetric Macdonald polynomials* of type  $A_{n-1}$ . The first property (1.14) implies in particular that  $E_\lambda(x)$  is homogeneous of degree  $|\lambda| = \sum_{i=1}^n \lambda_i$ , and is eventually a polynomial in  $x$  if  $\lambda \in L$ . Note also that the second property (1.15) is equivalent to saying that

$$(1.16) \quad f(Y)E_\lambda(x) = E_\lambda(x)f(q^\lambda t^{\rho(\lambda)})$$

for *any* Laurent polynomial  $f(Y)$  of the  $q$ -Dunkl operators. It is easy to see that the nonsymmetric Macdonald polynomials  $E_\lambda(x)$  ( $\lambda \in P$ ) actually have coefficients in  $\mathbb{Q}(q, t)$ . We also remark that, as a function of  $t$ , each  $E_\lambda(x) = E_\lambda(x|q, t)$  is regular at  $t = q^k$  ( $k = 0, 1, 2, \dots$ ). These polynomials  $E_\lambda(x)$  form a  $\mathbb{K}$ -basis of the ring  $\mathbb{K}[x^{\pm 1}]$  of Laurent polynomials or of the ring  $\mathbb{K}[x]$  of polynomials as follows:

$$(1.17) \quad \mathbb{K}[x^{\pm 1}] = \bigoplus_{\lambda \in P} \mathbb{K}E_\lambda(x), \quad \mathbb{K}[x] = \bigoplus_{\lambda \in L} \mathbb{K}E_\lambda(x).$$

It is known by [Ma1] that, for any dominant integral weight  $\lambda \in P^+$ , Macdonald's symmetric polynomial  $P_\lambda(x)$  is expressed as a linear combination of nonsymmetric Macdonald polynomials  $E_\mu(x)$  ( $\mu \in W\lambda$ ). To be more explicit, one has

$$(1.18) \quad P_\lambda(x) = \sum_{\mu \in W\lambda} a_{\lambda\mu} E_\mu(x) \quad \text{with} \quad a_{\lambda\mu} = \prod_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, \mu \rangle < 0}} \frac{1 - q^{\langle \alpha, \mu \rangle} t^{\langle \alpha, \rho(\mu) \rangle - 1}}{1 - q^{\langle \alpha, \mu \rangle} t^{\langle \alpha, \rho(\mu) \rangle}}.$$

We also give a remark on the action of the Hecke algebra  $H(\mathfrak{S}_n)$  on nonsymmetric Macdonald polynomials: For each  $i = 1, \dots, n-1$ , one has

$$(1.19) \quad T_i E_\mu(x) = t^{\frac{1}{2}} E_\mu(x) \quad \text{if} \quad \langle \alpha_i, \mu \rangle = 0,$$

and

$$(1.20) \quad T_i E_\mu(x) = u_{i,\mu} E_\mu(x) + v_{i,\mu} E_{s_i \mu}(x) \quad \text{if} \quad \langle \alpha_i, \mu \rangle \neq 0.$$

The coefficients  $u_{i,\mu}$ ,  $v_{i,\mu}$  are given by

$$(1.21) \quad u_{i,\mu} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle}}, \quad v_{i,\mu} = t^{\frac{1}{2}}$$

if  $\langle \alpha_i, \mu \rangle < 0$ , and

$$(1.22) \quad u_{i,\mu} = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle}},$$

$$v_{i,\mu} = t^{-\frac{1}{2}} \frac{(1 - q^{\langle \alpha_i, \mu \rangle} t^{\langle \alpha_i, \rho(\mu) \rangle + 1})(1 - q^{\langle \alpha_i, \mu \rangle} t^{\langle \alpha_i, \rho(\mu) \rangle - 1})}{(1 - q^{\langle \alpha_i, \mu \rangle} t^{\langle \alpha_i, \rho(\mu) \rangle})^2}$$

if  $\langle \alpha_i, \mu \rangle > 0$ .

## §2: Formula of Cauchy type.

It is well-known that the Macdonald polynomials  $P_\lambda(x|q, t)$  ( $\lambda \in L^+$ ) have the following formula of Cauchy type [Ma2]. Let now  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be two sets of variables, and define the function  $\Pi(x; y|q, t)$  by

$$(2.1) \quad \Pi(x; y|q, t) = \prod_{1 \leq i, j \leq n} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty},$$

where  $(x; q)_\infty = \prod_{i=0}^\infty (1 - q^i x)$ . The infinite products may be understood either in the sense of formal power series in appropriate variables, or in the analytic sense by assuming that  $q$  is a complex number with  $0 < |q| < 1$ . Then we have

$$(2.2) \quad \Pi(x; y|q, t) = \sum_{\lambda \in L^+} b_\lambda(q, t) P_\lambda(x|q, t) P_\lambda(y|q, t),$$

where the coefficients are given by

$$(2.3) \quad b_\lambda(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}} \quad (\lambda \in L^+)$$

in terms of the arm-length  $a(s) = \lambda_i - j$  and the leg-length  $l(s) = \lambda'_j - i$  of each box  $s = (i, j)$  in the partition  $\lambda$ .

We now introduce the function  $E(x; y|q, t)$  by setting

$$(2.4) \quad E(x; y|q, t) = \prod_{1 \leq j < i \leq n} \frac{(qtx_i y_j; q)_\infty}{(qx_i y_j; q)_\infty} \prod_{1 \leq i \leq n} \frac{(qtx_i y_i; q)_\infty}{(x_i y_i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}.$$

Note that this function can be factored as follows:

$$(2.5) \quad E(x; y|q, t) = \Pi(x; y|q, t) \prod_{i=1}^n \frac{1}{1 - tx_i y_i} \prod_{j < i} \frac{1 - x_i y_j}{1 - tx_i y_j}.$$

The ratio  $E(x; y|q, t) \Pi(x; y|q, t)^{-1}$  is essentially one of the rational functions (before symmetrization) which are used in [Mi1] and [KN].

**Theorem 2.1.** *The function  $E(x; y|q, t)$  has an expansion*

$$(2.6) \quad E(x; y|q, t) = \sum_{\lambda \in L} a_\lambda(q, t) E_\lambda(x|q, t) E_\lambda(y|q^{-1}, t^{-1}) \quad (a_\lambda(q, t) \in \mathbb{Q}(q, t))$$

summed over all compositions  $\lambda \in L$ .

In order to describe the coefficients  $a_\lambda(q, t)$  ( $\lambda \in L$ ) in the expansion (2.6), we use the notion of leg-length generalized to compositions, due to Knop and Sahi [KS]. For each box  $s = (i, j)$  in a composition  $\lambda \in L$ , the *generalized leg-length*  $l(s) = l_\lambda(s)$  is defined to be the sum

$$(2.7) \quad l(s) = l_{\text{up}}(s) + l_{\text{low}}(s)$$

of the upper and the lower leg-length, where

$$(2.8) \quad l_{\text{low}}(s) = \#\{k > i; j \leq \lambda_k \leq \lambda_i\}, \quad l_{\text{up}}(s) = \#\{k < i; j \leq \lambda_k + 1 \leq \lambda_i\}.$$

Note that this  $l(s)$  is the same as the usual leg-length if  $\lambda$  is a partition.

**Theorem 2.2.** For each composition  $\lambda \in L$ , the coefficient  $a_\lambda(q, t)$  in expansion (2.6) is given by

$$(2.9) \quad a_\lambda(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}} \quad (\lambda \in L),$$

where  $l(s) = l_\lambda(s)$  ( $s \in \lambda$ ) stands for the generalized leg-length in  $\lambda$ .

In this section, we will give a proof of Theorem 2.1. Also, we will describe the ratio of  $a_\lambda(q, t)$  and  $a_\mu(q, t)$  when  $\lambda$  is a partition and  $\mu$  is a composition in the orbit  $W\lambda$ . Theorem 2.2 will be established in two ways in Sections 3 and 4, by determining  $a_\lambda(q, t)$  for partitions  $\lambda \in L^+$ .

In what follows, we set  $K(x; y|q, t) = E(x; y^{-1}|q, t)$ , i.e.

$$(2.10) \quad K(x; y|q, t) = \prod_{1 \leq j < i \leq n} \frac{(qtx_i/y_j; q)_\infty}{(qx_i/y_j; q)_\infty} \prod_{1 \leq i \leq n} \frac{(qtx_i/y_i; q)_\infty}{(x_i/y_i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(tx_i/y_j; q)_\infty}{(x_i/y_j; q)_\infty}.$$

**Proposition 2.3.** For each  $i = 1, \dots, n$ , one has

$$(2.11) \quad Y_{i,x} K(x; y|q, t) = (Y_{i,y}^*)^{-1} K(x; y|q, t),$$

where the suffix  $x$  or  $y$  indicates the variables on which the operator should act, and  $Y_i^*$  is the dual  $q$ -Dunkl operator (cf. [KN]) defined by

$$(2.12) \quad Y_i^* = T_i^{-1} \cdots T_{n-1}^{-1} \omega T_1 \cdots T_{i-1}.$$

This proposition is a direct consequence of the following lemma.

**Lemma 2.4.**

- (1)  $T_{i,x} K(x; y|q, t) = T_{i,y} K(x; y|q, t) \quad (i = 1, \dots, n-1).$
- (2)  $\omega_x K(x; y|q, t) = \omega_y^{-1} K(x; y|q, t).$

Note that the function  $K(x; y|q, t)$  can be factored as follows:

$$(2.13) \quad K(x; y|q, t) = \Pi(x; y^{-1}|q, t) \psi(x, y), \quad \psi(x, y) = \prod_{i=1}^n \left( \frac{1}{1 - tx_i/y_i} \prod_{j < i} \frac{1 - x_i/y_j}{1 - tx_i/y_j} \right).$$

Since  $\Pi(x; y^{-1}|q, t)$  is symmetric both in  $x$  and in  $y$ , the formula of Lemma 2.4.(1) is equivalent to

$$(2.14) \quad T_{i,x} \psi(x, y) = T_{i,y} \psi(x, y) \quad (i = 1, \dots, n-1).$$

For a fixed  $i$ , it reduces to the identity

$$(2.15) \quad T_{i,x} \psi_{i,i+1}(x, y) = T_{i,y} \psi_{i,i+1}(x, y)$$

for

$$(2.16) \quad \psi_{i,i+1}(x, y) = \frac{1 - x_{i+1}/y_i}{(1 - tx_i/y_i)(1 - tx_{i+1}/y_i)(1 - tx_{i+1}/y_{i+1})},$$

which can be checked by a direct computation. (This computation is essentially contained in [Mil], [MN]). The formula of Lemma 2.4.(2) can be proved directly by chasing the arguments of  $q$ -shift factorials under the action  $\omega_y \omega_x$ .

*Proof of Theorem 2.1.* We begin with expanding the function  $E(x; y|q, t)$  in the form

$$(2.17) \quad E(x; y|q, t) = \sum_{\lambda \in L} E_\lambda(x|q, t) F_\lambda(y|q, t) \quad (F_\lambda(y|q, t) \in \mathbb{Q}(q, t)[y]).$$

We will show that each  $F_\lambda(x|q, t)$  is a constant multiple of  $E_\lambda(y|q^{-1}, t^{-1})$ . Note that Proposition 2.3 implies

$$(2.18) \quad Y_x^{-\mu} K(x; y|q, t) = (Y_y^*)^\mu K(x; y|q, t)$$

for any  $\mu \in P$ . Since

$$(2.19) \quad K(x; y|q, t) = \sum_{\lambda \in L} E_\lambda(x|q, t) F_\lambda(y^{-1}|q, t),$$

we have

$$(2.20) \quad Y_y^{*\mu} F_\lambda(y^{-1}|q, t) = F_\lambda(y^{-1}|q, t) q^{-\langle \mu, \lambda \rangle} t^{-\langle \mu, \rho(\lambda) \rangle}.$$

As is shown in [KN], for each  $i = 1, \dots, n$  the  $q$ -Dunkl operator  $Y_i$  and its dual  $Y_i^*$  are interchanged by the involution  $\iota$  on  $\mathbb{K}[y]$  such that  $\iota(y_j) = y_j^{-1}$  ( $j = 1, \dots, n$ ),  $\iota(q) = q^{-1}$ ,  $\iota(t^{\frac{1}{2}}) = t^{-\frac{1}{2}}$ . Hence we have

$$(2.21) \quad Y_y^\mu F_\lambda(y|q^{-1}, t^{-1}) = F_\lambda(y|q^{-1}, t^{-1}) q^{\langle \mu, \lambda \rangle} t^{\langle \mu, \rho(\lambda) \rangle}$$

for all  $\mu \in P$ . This implies that  $F_\lambda(y|q^{-1}, t^{-1})$  is a constant multiple of  $E_\lambda(y|q, t)$ . Namely,  $F_\lambda(y|q, t)$  is a constant multiple of  $E_\lambda(y|q^{-1}, t^{-1})$ .  $\square$

Before determining the coefficients  $a_\lambda(q, t)$ , we will describe the relation between  $a_\lambda(q, t)$  and  $a_\mu(q, t)$  when  $\lambda$  is dominant and  $\mu$  is in the orbit  $W\lambda$ .

**Lemma 2.5.** *If  $\lambda \in L^+$ ,  $\mu \in L$  and  $\mu \in W\lambda$ , then one has*

$$(2.22) \quad a_\mu(q, t) = a_\lambda(q, t) \prod_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, \mu \rangle < 0}} \frac{(1 - q^{-\langle \alpha, \mu \rangle} t^{-\langle \alpha, \rho(\mu) \rangle + 1})(1 - q^{-\langle \alpha, \mu \rangle} t^{-\langle \alpha, \rho(\mu) \rangle - 1})}{(1 - q^{-\langle \alpha, \mu \rangle} t^{-\langle \alpha, \rho(\mu) \rangle})^2}$$

*Proof.* By Lemma 2.4.(1), we have

$$(2.23) \quad \sum_{\mu \in L} a_\mu T_{i,x} E_\mu(x) \iota(E_\mu(y)) = \sum_{\mu \in L} a_\mu E_\mu(x) T_{i,y} \iota(E_\mu(y)),$$

for each  $i = 1, \dots, n-1$ , where  $a_\mu = a_\mu(q, t)$ . As we remarked at the end of Section 2, for each  $\mu \in L$ , we have  $T_{i,x} E_\mu(x) = t^{\frac{1}{2}} E_\mu(x)$  if  $\langle \alpha_i, \mu \rangle = 0$ , and

$$(2.24) \quad T_{i,x} E_\mu(x) = u_{i,\mu} E_\mu(x) + v_{i,\mu} E_{s_i \mu}(x).$$



when  $\langle \alpha_i, \mu \rangle \neq 0$ . Since  $T_{i,y}\iota(E_\mu(y)) = \iota(T_{i,y}^{-1}E_\mu(y))$ , and  $T_{i,y}^{-1} = T_{i,y} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ , we can determine the action of  $T_{i,y}$  on  $\iota(E_\mu(y))$  as follows:  $T_{i,\mu}\iota(E_\mu(y)) = t^{\frac{1}{2}}\iota(E_\mu(y))$  if  $\langle \alpha_i, \mu \rangle = 0$ , and

$$(2.25) \quad T_{i,y}\iota(E_\mu(y)) = (\iota(u_{i,\mu}) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}))\iota(E_\mu(y)) + \iota(v_{i,\mu})\iota(E_{s_i\mu}(y))$$

if  $\langle \alpha_i, \mu \rangle \neq 0$ . By substituting these formulas into (2.23), we obtain the recurrence formulas

$$(2.26) \quad a_{s_i\mu}v_{s_i\mu} = a_\mu\iota(v_\mu)$$

for  $\mu \in L$  with  $\langle \alpha_i, \mu \rangle \neq 0$ . Hence, by (2.21) and (2.22), we have

$$(2.27) \quad a_\mu = a_{s_i\mu} \frac{(1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle + 1})(1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle - 1})}{(1 - q^{-\langle \alpha_i, \mu \rangle} t^{-\langle \alpha_i, \rho(\mu) \rangle})^2}$$

for  $\mu \in L$  with  $\langle \alpha_i, \mu \rangle < 0$ . Assume now that  $\lambda \in L^+$  is a partition and  $\mu \in W.\lambda$ . Then one can find a sequence of simple roots  $\alpha_{j_1}, \dots, \alpha_{j_p}$  such that  $\mu = s_{j_1} \cdots s_{j_p}(\lambda)$  and that

$$(2.28) \quad \langle \alpha_{j_1}, \mu \rangle < 0, \langle \alpha_{j_2}, s_{j_1}(\mu) \rangle < 0, \dots, \langle \alpha_{j_p}, s_{j_{p-1}} \cdots s_{j_1}(\mu) \rangle < 0.$$

Note also that

$$(2.29) \quad \{\alpha \in \Delta^+ ; \langle \alpha, \mu \rangle < 0\} = \{\alpha_{j_1}, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} \cdots s_{j_{p-1}}(\alpha_{j_p})\}.$$

Applying formula (2.27) to  $\mu^{(0)} = \mu, \mu^{(1)} = s_{j_1}(\mu), \dots, \mu^{(p)} = s_{j_p} \cdots s_{j_1}(\mu) = \lambda$  repeatedly, we obtain formula (2.22) since  $\langle \alpha_{j_r}, \mu^{(r-1)} \rangle = \langle s_{j_1} \cdots s_{j_{r-1}}(\alpha_{j_r}), \mu \rangle$  and  $\langle \alpha_{j_r}, \rho(\mu^{(r-1)}) \rangle = \langle s_{j_1} \cdots s_{j_{r-1}}(\alpha_{j_r}), \rho(\mu) \rangle$  for  $r = 1, \dots, p$ .  $\square$

Lemma 2.5 can be rewritten in the combinatorial language. Imitating Sahi's notation [S], we set

$$(2.30) \quad d_\mu(q, t) = \prod_{s \in \mu} (1 - q^{a(s)+1} t^{l(s)+1}), \quad d'_\mu(q, t) = \prod_{s \in \mu} (1 - q^{a(s)+1} t^{l(s)})$$

for each  $\mu \in L$ . In this notation, Theorem 2.2 is equivalent to the equality  $a_\mu(q, t) = d_\mu(q, t)/d'_\mu(q, t)$ .

**Lemma 2.6.** *If  $\lambda \in L^+$ ,  $\mu \in L$  and  $\mu \in W.\lambda$ , then formula (2.22) has an alternative expression*

$$(2.31) \quad a_\mu(q, t) = a_\lambda(q, t) \frac{d'_\lambda(q, t) d_\mu(q, t)}{d_\lambda(q, t) d'_\mu(q, t)}.$$

*Proof.* For each  $\mu \in L$ , set  $a'_\mu = a'_\mu(q, t) = d_\mu(q, t)/d'_\mu(q, t)$ . For the proof of formula (2.31), it is enough to show

$$(2.32) \quad a_\mu(q, t) = a_{s_i\mu}(q, t) \frac{a'_\mu(q, t)}{a'_{s_i\mu}(q, t)}$$

assuming that  $\langle \alpha_i, \mu \rangle < 0$  ( $i = 1, \dots, n-1$ ); one can use (2.32) repeatedly to prove (2.31) in view of the expression  $\mu = s_{j_1} \cdots s_{j_p}(\lambda)$  as in the proof of Lemma 2.5. When  $\langle \alpha_i, \mu \rangle < 0$ , it is easy to see that the only difference between  $a'_\mu(q, t)$  and  $a'_{s_i \mu}(q, t)$  arises at the box  $s = (i+1, \mu_i+1) \in \mu$  (or at  $s' = (i, \mu_i+1) \in s_i \mu$ ). In fact we have

$$(2.33) \quad a'_\mu = a'_{s_i \mu} \frac{(1 - q^{\mu_{i+1} - \mu_i} t^{l_\mu(s)+1})(1 - q^{\mu_{i+1} - \mu_i} t^{l_\mu(s)-1})}{(1 - q^{\mu_{i+1} - \mu_i} t^{l_\mu(s)})^2}.$$

On the other hand, one can directly check that

$$(2.34) \quad -\langle \alpha_i, \mu \rangle = \mu_{i+1} - \mu_i, \quad -\langle \alpha_i, \rho(\mu) \rangle = l_\mu(s)$$

by the definition (1.13) of  $\rho(\mu)$ . Hence we have (2.32) by comparing (2.27) and (2.33).  $\square$

### §3: First proof of Theorem 2.2.

In this section, we calculate the coefficients  $a_\lambda = a_\lambda(q, t)$  for partitions  $\lambda \in L^+$  by means of asymptotic analysis of a  $q$ -Selberg type integral. Such an argument has been employed in [Mi2] to evaluate the scalar product for Macdonald's symmetric polynomials.

We now assume that  $q$  and  $t$  are complex numbers with  $0 < |q|, |t| < 1$ , and recall Cherednik's scalar product [C1]. For  $f = f(x|q, t), g = g(x|q, t) \in \mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$ , we define

$$(3.1) \quad (f, g) = \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} f(x|q, t) g(x^{-1}|q^{-1}, t^{-1}) w(x|q, t) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n},$$

where

$$(3.2) \quad w(x|q, t) = \prod_{1 \leq i < j \leq n} \frac{(x_i/x_j; q)_\infty (qx_j/x_i; q)_\infty}{(tx_i/x_j; q)_\infty (qtx_j/x_i; q)_\infty}$$

and  $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n; |x_i| = 1 \ (1 \leq i \leq n)\}$  with the standard orientation. Note that

$$(3.3) \quad w(x|q, q^k) = \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_k (qx_j/x_i; q)_k$$

if  $t = q^k$  ( $k = 0, 1, 2, \dots$ ), where  $(a; q)_k = (a; q)_\infty / (at; q)_\infty = \prod_{0 \leq i \leq k-1} (1 - aq^i)$ . The nonsymmetric Macdonald polynomials  $E_\lambda(x; q, t)$  ( $\lambda \in L$ ) form an orthogonal basis of  $\mathbb{K}[x]$  with respect to this scalar product:

$$(3.4) \quad (E_\lambda, E_\mu) = 0 \quad \text{if } \lambda \neq \mu.$$

It is known furthermore that, if  $t = q^k$  ( $k \in \mathbb{N}$ ) and  $\lambda \in L^+$  is a partition, then

$$(3.5) \quad (E_\lambda, E_\lambda) = \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i - \lambda_j + 1 + k(j-i)}; q)_k}{(q^{\lambda_i - \lambda_j + 1 + k(j-i-1)}; q)_k}.$$

(See [Ma1], [C1].) For general values of  $t$  with  $|t| < 1$ , one has

$$(3.6) \quad (E_\lambda, E_\lambda) = \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i - \lambda_j + 1} t^{j-i}; q)_\infty^2}{(q^{\lambda_i - \lambda_j + 1} t^{j-i+1}; q)_\infty (q^{\lambda_i - \lambda_j + 1} t^{j-i-1}; q)_\infty}$$

for any partition  $\lambda \in L^+$ . Note that, as functions in  $t$ , the both sides of (3.6) are meromorphic in  $\{|t| < 1\}$ . Since this formula is valid at the points  $t = q^k$  ( $k \in \mathbb{N}$ ) accumulating at the origin, one can conclude that the left hand side of (3.6) is eventually holomorphic near  $t = 0$ , and that (3.6) is valid as an equality of analytic functions. It is also known that, if  $\mu \in W.\lambda$  is a composition in the  $W$ -orbit of a partition  $\lambda$ , then we have

$$(3.7) \quad \frac{(E_\mu, E_\mu)}{(E_\lambda, E_\lambda)} = \prod_{\substack{\alpha \in \Delta^+ \\ \langle \alpha, \mu \rangle < 0}} \frac{(1 - q^{-\langle \alpha, \mu \rangle} t^{-\langle \alpha, \rho(\mu) \rangle})^2}{(1 - q^{-\langle \alpha, \mu \rangle} t^{-\langle \alpha, \rho(\mu) \rangle + 1})(1 - q^{-\langle \alpha, \mu \rangle} t^{-\langle \alpha, \rho(\mu) \rangle - 1})}.$$

We now consider  $x = (x_1, \dots, x_n)$  as variables inside the polydisc  $\{|x_j| < 1; j = 1, \dots, n\} \subset \mathbb{C}^n$ . Note that the series expansion

$$(3.8) \quad K(x; y|q, t) = \sum_{\mu \in L} a_\mu(q, t) E_\mu(x|q, t) E_\mu(y^{-1}|q^{-1}, t^{-1}),$$

in Theorem 2.1 is then uniformly convergent on  $\mathbb{T}^n$  in  $y$ . Hence by the orthogonality relations (3.4) we have

$$(3.9) \quad \begin{aligned} & \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} K(x; y|q, t) E_\lambda(y|q, t) w(y|q, t) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} \\ &= \sum_{\mu \in L} a_\mu(q, t) E_\mu(x|q, t) (E_\lambda, E_\mu) \\ &= a_\lambda(q, t) E_\lambda(x|q, t) (E_\lambda, E_\lambda). \end{aligned}$$

We study the asymptotic behavior of the left hand side of (3.9) in  $x$  in the region

$$(3.10) \quad 1 > |x_1| \gg |x_2| \gg \cdots \gg |x_n|.$$

For the moment, we assume that  $t = q^k$  ( $k = 0, 1, 2, \dots$ ) and that  $\lambda$  is a partition.

**Proposition 3.1.** *If  $t = q^k$  ( $k \in \mathbb{N}$ ) and  $\lambda \in L^+$ , then one has*

$$(3.11) \quad \begin{aligned} & \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} K(x; y|q, q^k) E_\lambda(y|q, q^k) w(y|q, q^k) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} \\ & \sim x^\lambda \prod_{i=1}^n \frac{(q^{\lambda_i + (n-i)k+1}; q)_k}{(q; q)_k}, \end{aligned}$$

as  $\max\{|x_2/x_1|, |x_3/x_2|, \dots, |x_n/x_{n-1}|\}$  tends to 0.

Proposition will be proved later in this section.

We apply Proposition 3.1 to compare the coefficients of  $x^\lambda$  in equality (3.9). Since we know by (3.9) that the integral has the asymptotic behavior  $x^\lambda a_\lambda(q, t)(E_\lambda, E_\lambda) + \dots$ , we obtain

$$(3.12) \quad a_\lambda(q, q^k)(E_\lambda, E_\lambda) = \prod_{i=1}^n \frac{(q^{\lambda_i + (n-i)k+1}; q)_k}{(q; q)_k}$$

for any partition  $\lambda \in L^+$ , provided that  $t = q^k$  ( $k \in \mathbb{N}$ ). This is equivalent to

$$(3.13) \quad \begin{aligned} a_\lambda(q, q^k) &= \prod_{1 \leq i \leq j \leq n} \frac{(q^{\lambda_i - \lambda_{j+1} + 1 + k(j-i)}; q)_k}{(q^{\lambda_i - \lambda_j + 1 + k(j-i)}; q)_k} \\ &= \prod_{i=1}^n \prod_{j=i}^n \frac{(q^{\lambda_i - \lambda_j + 1 + k(j-i+1)}; q)_{\lambda_j - \lambda_{j+1}}}{(q^{\lambda_i - \lambda_j + 1 + k(j-i)}; q)_{\lambda_j - \lambda_{j+1}}} \end{aligned}$$

by formula (3.5), where we set  $\lambda_{n+1} = 0$ . By analytic continuation as before, we get

$$(3.14) \quad a_\lambda(q, t) = \prod_{i=1}^n \prod_{j=i}^n \frac{(q^{\lambda_i - \lambda_j + 1} t^{j-i+1}; q)_{\lambda_j - \lambda_{j+1}}}{(q^{\lambda_i - \lambda_j + 1} t^{j-i}; q)_{\lambda_j - \lambda_{j+1}}},$$

since  $a_\lambda(q, t)$  is a rational function in  $t$ . Formula (3.14) implies that  $a_0 = 1$  and that

$$(3.15) \quad a_{\lambda+(1^m)}(q, t) = a_\lambda(q, t) \prod_{i=1}^m \frac{1 - q^{\lambda_i+1} t^{m-i+1}}{1 - q^{\lambda_i+1} t^{m-i}}$$

for any  $\lambda \in L^+$  with  $l(\lambda) \leq m$ . In fact, the difference between  $a_\lambda(q, t)$  and  $a_{\lambda+(1^m)}(q, t)$  appears only in the factors in (3.14) with  $1 \leq i \leq m$  and  $j = m$ . From (3.15) it follows immediately that

$$(3.16) \quad a_\lambda(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}} = \frac{d_\lambda(q, t)}{d'_\lambda(q, t)}$$

for all  $\lambda \in L^+$ . Hence by Lemma 2.6, the same formula (3.16) holds for all compositions  $\lambda \in L$  if  $l(s)$  is understood as the generalized leg-length. This completes the proof of Theorem 2.2.

Note that formula (3.12) extends to the equality

$$(3.17) \quad a_\lambda(q, t)(E_\lambda, E_\lambda) = \prod_{i=1}^n \frac{(q^{\lambda_i+1} t^{n-i}; q)_\infty (qt; q)_\infty}{(q^{\lambda_i+1} t^{n-i+1}; q)_\infty (q; q)_\infty} \quad (\lambda \in L^+)$$

of analytic functions in  $t$ . On the other hand, by comparing formula (2.22) of Lemma 2.5 and (3.7), we see that

$$(3.18) \quad a_\mu(q, t)(E_\mu, E_\mu) = a_\lambda(q, t)(E_\lambda, E_\lambda)$$

for all compositions  $\mu \in W.\lambda$ . Summarizing these remarks, we obtain the following theorem.

**Theorem 3.2.** *For each composition  $\lambda \in \mathbb{N}^n$ , we have*

$$(3.19) \quad \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} K(x; y|q, t) E_\lambda(y|q, t) w(y|q, t) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} = C_\lambda \cdot E_\lambda(x|q, t)$$

for  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$  with  $|x_j| < 1$  ( $j = 1, \dots, n$ ). Here  $C_\lambda$  is constant on each  $W$ -orbit, and is given by

$$(3.20) \quad C_\lambda = \left( \frac{(qt; q)_\infty}{(q; q)_\infty} \right)^n \prod_{i=1}^n \frac{(q^{\lambda_i+1} t^{n-i}; q)_\infty}{(q^{\lambda_i+1} t^{n-i+1}; q)_\infty}$$

when  $\lambda \in L^+$  is a partition.

In this sense, our function  $K(x; y|q, t)$  is a reproducing kernel for nonsymmetric Macdonald polynomials.

In the rest of this section, we will prove Proposition 3.1. From now on we assume that  $t = q^k$  for a fixed  $k \in \mathbb{N}$ , and omit  $(q, t) = (q, q^k)$  in the notation unless it might lead to confusion.

In proving Proposition, we may assume that  $0 < |x_j| < 1$  for  $j = 1 \dots, n$  and that all  $x_j$ 's are mutually distinct. Recall that

$$(3.21) \quad \begin{aligned} K(x; y) &= \prod_{j < i} \frac{1}{(qx_i/y_j; q)_k} \prod_i \frac{1}{(x_i/y_i; q)_{k+1}} \prod_{i < j} \frac{1}{(x_i/y_j; q)_k} \\ &= \prod_{i, j} \frac{1}{(q^{\theta(i > j)} x_i/y_j; q)_{k+\delta_{ij}}}, \end{aligned}$$

where  $\theta(i > j) = 1$  if  $i > j$ , and  $\theta(i > j) = 0$  if  $i \leq j$ , and

$$(3.22) \quad w(y) = \prod_{i < j} (y_i/y_j; q)_k (qy_j/y_i)_k.$$

Note first that, as a function of  $y_j$  ( $1 \leq j \leq n$ ), the integrand

$$(3.23) \quad K(x; y) E_\lambda(y) w(y) (y_1 \cdots y_n)^{-1}$$

of (3.11) is regular at  $y_j = 0$  and has poles only at  $y_j = x_s q^l$  ( $1 \leq s \leq n, l \in \mathbb{N}$ ). The range of  $l$  can be specified as follows:

$$(3.24) \quad \begin{array}{lll} (1) & 0 \leq l < k & \text{if } 1 \leq s < j, \\ (2) & 0 \leq l \leq k & \text{if } s = j, \\ (3) & 0 < l \leq k & \text{if } j < s \leq n. \end{array} \quad \text{and}$$

The integral (3.11) will be computed by picking up successively the residues at  $y_j = x_s q^l$  ( $1 \leq j, s \leq n$ ) with  $l$  satisfying (3.24). To make clear this inductive process, we propose a lemma.

For any pair  $(I, J)$  of subsets of  $\{1, \dots, n\}$  with  $|I| = |J|$ , we extend the notation of (3.21), (3.23) as follows:

$$(3.25) \quad K(x_I; y_J) = \prod_{\substack{i \in I \\ j \in J}} \frac{1}{(q^{\theta(i > j)} x_i/y_j; q)_{k+\delta_{ij}}}, \quad w(y_J) = \prod_{\substack{i, j \in J \\ i < j}} (y_i/y_j; q)_k (qy_j/y_i)_k,$$

where  $x_I = (x_i)_{i \in I}$  and  $y_J = (y_j)_{j \in J}$ .

**Lemma 3.3.** Fix two indices  $j \in J$ ,  $s \in I$  and  $l \in \mathbb{N}$  satisfying (3.24), and set  $J' = J \setminus \{j\}$  and  $I' = I \setminus \{s\}$ . Then the residue

$$(3.26) \quad f(x_I; y_{J'}) = \text{Res}_{y_j = x_s q^l} (K(x_I; y_J) w(y_J) \frac{dy_j}{y_j})$$

at  $y_j = x_s q^l$  has an expression

$$(3.27) \quad f(x_I; y_{J'}) = g(x_I; y_{J'}) K(x_{I'}; y_{J'}) w(y_{J'}),$$

where  $g(x_I; y_{J'})$  is a polynomial in  $y_{J'}$  with coefficients in  $\mathbb{K}(x_I)$ .

*Proof.* The only factors to be checked are

$$(3.28) \quad \frac{(q^{-l} y_\mu / x_s; q)_k (q^{l+1} x_s / y_\mu; q)_k}{(q^{\theta(s > \mu)} x_s / y_\mu; q)_{k + \delta_{s\mu}}}$$

for  $\mu \in J'$  with  $\mu < j$ , and

$$(3.29) \quad \frac{(q^l x_s / y_\mu; q)_k (q^{1-l} y_\mu / x_s; q)_k}{(q^{\theta(s > \mu)} x_s / y_\mu; q)_{k + \delta_{s\mu}}}$$

for  $\mu \in J'$  with  $\mu > j$ . As a function of  $y_\mu$ , the numerators of (3.28) and (3.29) have zeros at  $y_\mu = x_s q^m$  for  $m \in \{l - k + 1, l - k + 2, \dots, l + k\}$  and for  $m \in \{l - k, l - k + 1, \dots, l + k - 1\}$ , respectively. From this, it turns out that the rational functions (3.28) and (3.29) are in fact regular at  $y_\mu = x_s q^m$  for all  $m \in \mathbb{N}$ , provided that  $l$  satisfies the condition of (3.24).  $\square$

Let us now integrate  $K(x; y) E_\lambda(y) w(y) / y_1$  with respect to the variable  $y_1$ . Then we have

$$(3.30) \quad \begin{aligned} & \frac{1}{2\pi\sqrt{-1}} \int_{|y_1|=1} K(x; y) E_\lambda(y) w(y) \frac{dy_1}{y_1} \\ &= \sum_{s=1}^n \sum_{l=0}^k \text{Res}_{y_1 = x_s q^l} (K(x; y) E_\lambda(y) w(y) \frac{dy_1}{y_1}) \end{aligned}$$

since the integrand is regular at  $y_1 = 0$ . If we regard each summand of the right-hand side as a function of  $y_2$ , it has a zero at  $y_2 = 0$  and has poles only at  $y_2 = x_r q^m$  for  $r \neq s$ ,  $0 \leq m \leq k$  by Lemma 3.3. Hence we have

$$(3.31) \quad \begin{aligned} & \left( \frac{1}{2\pi\sqrt{-1}} \right)^2 \int_{|y_1|=|y_2|=1} K(x; y) E_\lambda(y) w(y) \frac{dy_1 dy_2}{y_1 y_2} \\ &= \sum_{\substack{1 \leq s, r \leq n \\ s \neq r}} \sum_{0 \leq l, m \leq k} \text{Res}_{\substack{y_1 = x_s q^l \\ y_2 = x_r q^m}} (K(x; y) E_\lambda(y) w(y) \frac{dy_1 dy_2}{y_1 y_2}). \end{aligned}$$

Applying Lemma 3.3 repeatedly from  $j = 1$  to  $n$ , we obtain the equality

$$(3.32) \quad \begin{aligned} & \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} K(x; y) E_\lambda(y) w(y) \frac{dy}{y} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \sum_{0 \leq l_1, \dots, l_n \leq k} \text{Res}_{y = (x_{\sigma(1)} q^{l_1}, \dots, x_{\sigma(n)} q^{l_n})} \left( K(x; y) E_\lambda(y) w(y) \frac{dy}{y} \right), \end{aligned}$$

where we used the abbreviation  $dy/y = dy_1 \cdots dy_n / y_1 \cdots y_n$ .

We now investigate the asymptotic behavior of this function in the region (3.10). Note that we have

$$(3.33) \quad \begin{aligned} & \text{Res}_{y=(x_{\sigma(1)}q^{l_1}, \dots, x_{\sigma(n)}q^{l_n})} \left( K(x; y) E_\lambda(y) w(y) \frac{dy}{y} \right) \\ &= \text{Res}_{y=(q^{l_1}, \dots, q^{l_n})} \left( K(x; x_\sigma y) E_\lambda(x_\sigma y) w(x_\sigma y) \frac{dy}{y} \right), \end{aligned}$$

where  $x_\sigma y$  stands for  $(x_{\sigma(1)}y_1, \dots, x_{\sigma(n)}y_n)$ . The function  $K(x; x_\sigma y)w(x_\sigma y)$  can be written in the following form:

$$(3.34) \quad \prod_{i,j} \frac{1}{(q^{\theta(i \geq j)} x_i / x_{\sigma(j)} y_j; q)_k} \prod_{i \neq j} (q^{\theta(i > j)} x_{\sigma(i)} y_i / x_{\sigma(j)} y_j; q)_k \times \prod_{j=1}^n \frac{1}{1 - x_j / x_{\sigma(j)} y_j}.$$

The product of the first two factors altogether is bounded in the limit (3.10). If  $\sigma \in \mathfrak{S}_n$  is *not* the identity element, one can take a suffix  $i$  such that  $i < \sigma(i)$ . As an effect of the factor  $1/(1 - x_i/x_{\sigma(i)}y_i)$  in the third factors of (3.34), we then have

$$(3.35) \quad |K(x; x_\sigma y)w(x_\sigma y)| = O\left(\left|\frac{x_{\sigma(i)}}{x_i}\right|\right).$$

Since  $\lambda$  is a partition,  $x^{-\lambda}E_\lambda(x_\sigma y)$  is also bounded in the limit (3.10). Hence we have

$$(3.36) \quad x^{-\lambda} \text{Res}_{y=(q^{l_1}, \dots, q^{l_n})} \left( K(x; x_\sigma y) E_\lambda(x_\sigma y) w(x_\sigma y) \frac{dy}{y} \right) = O\left(\left|\frac{x_{\sigma(i)}}{x_i}\right|\right),$$

provided that  $\sigma$  is not the identity element. If  $\sigma$  is the identity element, the function

$$(3.37) \quad \text{Res}_{y=(q^{l_1}, \dots, q^{l_n})} \left( K(x; xy) E_\lambda(xy) w(xy) \frac{dy}{y} \right)$$

tends to

$$(3.38) \quad \begin{aligned} & \text{Res}_{y=(q^{l_1}, \dots, q^{l_n})} \left( \prod_{i=1}^n \frac{y_i^{(n-i)k}}{(1/y_i; q)_{k+1}} \{ (x_1 y_1)^{\lambda_1} \cdots (x_n y_n)^{\lambda_n} + \cdots \} \frac{dy}{y} \right) \\ &= x^\lambda \text{Res}_{y=(q^{l_1}, \dots, q^{l_n})} \left( \prod_{i=1}^n \frac{y_i^{(n-i)k+\lambda_i}}{(1/y_i; q)_{k+1}} \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} \right) + \text{lower terms} \\ &= x^\lambda \prod_{i=1}^n \left\{ \frac{(q^{-k}; q)_{l_i}}{(q; q)_k (q; q)_{l_i}} (q^{\lambda_i + (n-i+1)k+1})^{l_i} \right\} + \text{lower terms}. \end{aligned}$$

Combining (3.32) with (3.36) and (3.38), we obtain

$$(3.39) \quad \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\mathbb{T}^n} K(x; y) E_\lambda(y) w(y) \frac{dy}{y} = C_\lambda x^\lambda + \text{lower terms}$$

in the region (3.10), with the leading coefficient

$$(3.40) \quad C_\lambda = \sum_{0 \leq l_1, \dots, l_n \leq k} \prod_{i=1}^n \left\{ \frac{(q^{-k}; q)_{l_i}}{(q; q)_k (q; q)_{l_i}} (q^{\lambda_i + (n-i+1)k+1})^{l_i} \right\}.$$

By using the  $q$ -binomial theorem

$$(3.41) \quad \sum_{l \geq 0} \frac{(a; q)_l}{(q; q)_l} z^l = \frac{(az; q)_\infty}{(z; q)_\infty} \quad (|z| < 1),$$

the constant  $C_\lambda$  is determined as

$$(3.42) \quad C_\lambda = \prod_{i=1}^n \frac{(q^{\lambda_i + (n-i)k+1}; q)_k}{(q; q)_k}.$$

This completes the proof of Proposition 3.1.

#### §4: Second proof of Theorem 2.2.

In this section, we will give a proof of Theorem 2.2 based on the principal specialization of nonsymmetric Macdonald polynomials, along the line similar to that in Sahi [S].

We begin with a lemma.

**Lemma 4.1.** *The function  $\prod_{i=1}^n (ux_i; q)_\infty / (x_i; q)_\infty$  has an expansion*

$$(4.1) \quad \prod_{i=1}^n \frac{(ux_i; q)_\infty}{(x_i; q)_\infty} = \sum_{\lambda \in L^+} P_\lambda(x|q, t) f_\lambda(u|q, t),$$

in terms of Macdonald polynomials. The coefficients are given by

$$(4.2) \quad f_\lambda(u|q, t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{-l'(s)} u}{1 - q^{a(s)+1} t^{l(s)}}$$

for each partition  $\lambda$ . Here, for each box  $s = (i, j)$  in  $\lambda$ ,  $a'(s) = j - 1$  and  $l'(s) = i - 1$  stand for the coarm-length and the coleg-length of  $s$  in  $\lambda$ , and  $n(\lambda) = \sum_{s \in \lambda} l(s) = \sum_{s \in \lambda} l'(s)$ .

*Proof.* Let  $m$  be an integer with  $m \geq n$  and take the variables  $y = (y_1, \dots, y_m)$ . Then we have

$$(4.3) \quad \prod_{i=1}^n \prod_{j=1}^m \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\lambda \in L^+} b_\lambda(q, t) P_\lambda(x|q, t) P_\lambda(y|q, t).$$

By the evaluation at  $y = (1, t, \dots, t^{m-1})$ , we get

$$(4.4) \quad \prod_{i=1}^n \frac{(t^m x_i; q)_\infty}{(x_i; q)_\infty} = \sum_{\lambda \in L^+} b_\lambda(q, t) P_\lambda(x|q, t) P_\lambda(1, t, \dots, t^{m-1}|q, t).$$



Hence we have

$$(4.5) \quad f_\lambda(t^m|q, t) = b_\lambda(q, t)P_\lambda(1, t, \dots, t^{m-1}|q, t).$$

It is known by [Ma2] that

$$(4.6) \quad P_\lambda(1, t, \dots, t^{m-1}|q, t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{m-l'(s)}}{1 - q^{a(s)} t^{l(s)+1}}.$$

Combining this with the formula for  $b_\lambda(q, t)$ , we obtain

$$(4.7) \quad f_\lambda(t^m|q, t) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{m-l'(s)}}{1 - q^{a(s)+1} t^{l(s)}}.$$

Since  $f_\lambda(u|q, t)$  is a polynomial in  $u$ , and  $m \geq n$  is arbitrary, we obtain the desired formula.  $\square$

We will prove Theorem 2.2 by evaluating of the function  $E(x; y|q, t)$  at the point  $y = t^\delta = (t^{n-1}, t^{n-2}, \dots, 1)$ . From the definition of  $E(x; y|q, t)$ , we easily see that

$$(4.8) \quad E(x; t^\delta|q, t) = \prod_{i=1}^n \frac{(qt^n x_i; q)_\infty}{(x_i; q)_\infty}.$$

By Lemma 4.1, we can expand this into the sum over all  $P_\lambda(x|q, t)$ :

$$(4.9) \quad E(x; t^\delta|q, t) = \sum_{\lambda \in L^+} P_\lambda(x|q, t) f_\lambda(qt^n|q, t).$$

For each partition  $\lambda \in L^+$ , Macdonald's symmetric polynomial  $P_\lambda(x|q, t)$  can be written as a linear combination of nonsymmetric ones  $E_\mu(x|q, t)$ , summed over all compositions  $\mu$  in the  $W$ -orbit  $W.\lambda$  (see [Ma1]) :

$$(4.10) \quad P_\lambda(x|q, t) = \sum_{\mu \in W.\lambda} a_{\lambda\mu}(q, t) E_\mu(x|q, t) \quad (a_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t))$$

with  $a_{\lambda\lambda}(q, t) = 1$ . Hence we have

$$(4.11) \quad E(x; t^\delta|q, t) = \sum_{\mu \in L} f_{\mu^+}(qt^n|q, t) a_{\mu^+ \mu}(q, t) E_\mu(x|q, t).$$

On the other hand, by Theorem 2.1.(1),  $E(x; y|q, t)$  has the expansion

$$(4.12) \quad E(x; y|q, t) = \sum_{\mu \in L} a_\mu(q, t) E_\mu(x|q, t) E_\mu(y|q^{-1}, t^{-1}).$$

Hence we have

$$(4.13) \quad E(x; t^\delta|q, t) = \sum_{\mu \in L} a_\mu(q, t) E_\mu(t^\delta|q^{-1}, t^{-1}) E_\mu(x|q, t).$$

Comparing the two expansions (4.11) and (4.13) of  $E(x; t^\delta | q, t)$ , we obtain

$$(4.14) \quad f_{\mu^+}(qt^n | q, t) a_{\mu^+ \mu}(q, t) = a_\mu(q, t) E_\mu(t^\delta | q^{-1}, t^{-1}).$$

In particular, if  $\lambda$  is a partition, then we have

$$(4.15) \quad f_\lambda(qt^n | q, t) = a_\lambda(q, t) E_\lambda(t^\delta | q^{-1}, t^{-1}).$$

Evaluation of nonsymmetric Macdonald polynomials at  $t^{-\delta}$  is already carried out by Cherednik [C2]. If  $\lambda \in L^+$  is a partition, the value  $E_\lambda(t^{-\delta} | q, t)$  can be rewritten as follows:

$$(4.16) \quad E_\lambda(t^{-\delta} | q, t) = t^{-(n-1)|\lambda|+n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)+1} t^{n-l'(s)}}{1 - q^{a(s)+1} t^{l(s)+1}}.$$

From this formula, we have

$$(4.17) \quad E_\lambda(t^\delta | q^{-1}, t^{-1}) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)+1} t^{n-l'(s)}}{1 - q^{a(s)+1} t^{l(s)+1}}.$$

Substituting (4.2) and (4.17) into (4.15), we have

$$(4.18) \quad a_\lambda(q, t) = \frac{f_\lambda(qt^n | q, t)}{E_\lambda(t^\delta | q^{-1}, t^{-1})} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}$$

for all partition  $\lambda \in L^+$ . Namely we have  $a_\lambda(q, t) = d_\lambda(q, t)/d'_\lambda(q, t)$  for all  $\lambda \in L^+$  with the notation of Section 2. Hence by Lemma 2.5 we have

$$(4.19) \quad a_\mu(q, t) = \frac{d_\mu(q, t)}{d'_\mu(q, t)} = \prod_{s \in \mu} \frac{1 - q^{a(s)+1} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}$$

for all composition  $\mu \in L$ . This completes the proof of Theorem 2.2.

*Remark 4.2.* In Section 3, we determined the coefficients  $a_\lambda(q, t)$  by combining asymptotic analysis of  $q$ -Selberg type integrals and the formulas for scalar products  $(E_\lambda, E_\lambda)$ . Since we have derived the formulas for  $a_\lambda(q, t)$  along a different route in this section, we can also use the argument of Section 3 conversely to determine the scalar products  $(E_\lambda, E_\lambda)$  (cf. [Mi2]).

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